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THE GOMPERTZ CURVE AS A GROWTH CURVE

BY CHARLES P. WINSOR

DEPARTMENT OF BIOLOGY, SCHOOL OF HYGIENE AND PUBLIC HEALTH, JOHNS HOPKINS
UNIVERSITY

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In 1825 Benjamin Gompertz published a paper in the *Philosophical Transactions of the Royal Society*, "On the Nature of the Function Expressive of the Law of Human Mortality," in which he showed that "if the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction," then the number of survivors at any age x would be given by the equation

$$L_x = kg^{e^x} \quad (1)$$

(It is clear that Gompertz means equal proportions, not equal absolute amounts, of the "power to oppose destruction.")

The Gompertz curve was for long of interest only to actuaries. More recently, however, it has been used by various authors as a growth curve, both for biological and for economic phenomena. It is the purpose of the present note to consider some of the mathematical properties of this curve, and to indicate to some extent its usefulness and its limitations as a growth curve.

For actual purposes the curve is generally written in the form (1); but for our purpose it is more convenient to write it

$$y = ke^{-e^{a-bx}} \quad (2)$$

in which k and b are essentially positive quantities.

From (2) it is clear that as x becomes negatively infinite y will approach zero, and as x becomes positively infinite y will approach k . Differentiating (2) we have

$$\frac{dy}{dx} = kbe^{a-bx} e^{-e^{a-bx}} = b ye^{a-bx} \quad (3)$$

and it is apparent that the slope is always positive for finite values of x , and approaches zero for infinite values of x . Differentiating again we have

$$\frac{d^2y}{dx^2} = b^2ye^{a-bx}(e^{a-bx} - 1). \quad (4)$$

From (4) we see that there will be a point of inflection when

$$x = \frac{a}{b}$$

The ordinate at the point of inflection is

$$y = \frac{k}{e}$$

or approximately, when 37 per cent of the final growth has been reached.

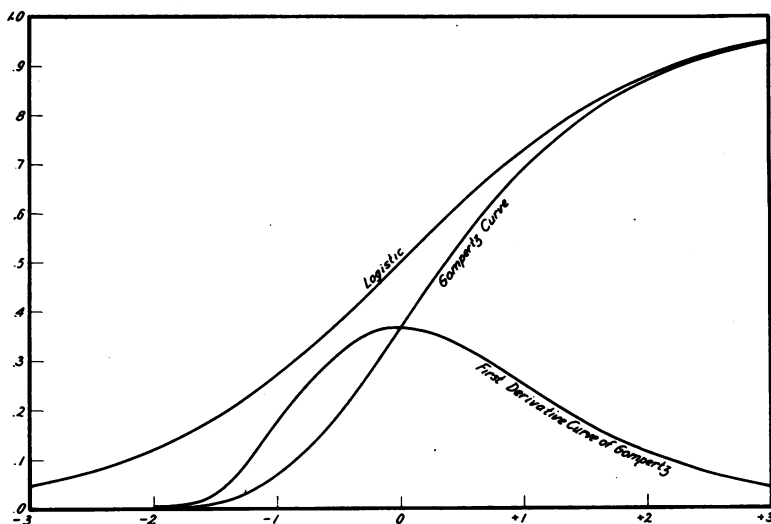


FIGURE 1

Figure 1 shows the form of the curve for the case $k = 1$, $a = 0$, $b = 1$; there are also shown the logistic and the first derivative of the Gompertz curve.

Equations:

Gompertz:
$$y = e^{-e^{-x}}$$

Logistic:
$$y = \frac{1}{1 + e^{-x}}$$

When we desire, therefore, to fit growth data which show a point of

inflection in the early part of the growth cycle, when approximately 35 to 40 per cent of the total growth has been realized, we may use the Gompertz curve with the expectation that the approximation to the data will be good. There seems, however, no particular reason to expect that the Gompertz curve will show any wider range of fitting power than any other three-constant S-shaped curve. For example, the logistic

$$y = \frac{k}{1 + e^{a-bx}} \quad (5)$$

possesses the same number of constants as the Gompertz curve, but has the point of inflection mid-way between the asymptotes. The degree of "skewness" in the Gompertz curve is just as fixed as in the logistic; and it is clear that to introduce a variable degree of skewness into a growth curve will require at least four constants.

To illustrate the mathematical properties of the Gompertz curve and the logistic, the following table has been prepared.

PROPERTY	GOMPERTZ	LOGISTIC
Equation	$y = ke^{-e^{-bx}}$	$y = \frac{k}{1 + e^{a-bx}}$
Number of constants	3	3
Asymptotes	$\begin{cases} y = 0 \\ y = k \end{cases}$	$\begin{cases} y = 0 \\ y = k \end{cases}$
Inflection	$\begin{cases} x = \frac{a}{b} \\ y = \frac{k}{e} \end{cases}$	$\begin{cases} x = \frac{a}{b} \\ y = \frac{k}{2} \end{cases}$
Straight line form of equation	$\log \log \frac{k}{y} = a - bx$	$\log \frac{k-y}{y} = a - bx$
Symmetry	Assymetrical	Symmetrical about inflection
Growth rate	$\frac{dy}{dx} = bye^{a-bx} = by \log \frac{k}{y}$	$\frac{dy}{dx} = \frac{b}{k} y(k-y)$
Maximum growth rate	$\frac{bk}{e}$	$\frac{bk}{4}$
Relative growth rate as function of time	$\frac{1}{y} \frac{dy}{dx} = be^{a-bx}$	$\frac{1}{y} \frac{dy}{dx} = \frac{b}{1 + e^{-a+bx}}$
Relative growth rate as function of size	$\frac{1}{y} \frac{dy}{dx} = b (\log k - \log y)$	$\frac{1}{y} \frac{dy}{dx} = \frac{b}{k} (k - y)$

The parallelism between the Gompertz curve and the logistic may be carried further. It has been found useful, for example, to add a constant term to the logistic, giving it a lower asymptote different from zero

$$y = d + \frac{k}{1 + e^{a-bx}} \quad (6)$$

Clearly this procedure is equally applicable to the Gompertz curve, giving

$$y = d + ke^{-e^{-bx}} \quad (7)$$

It is also clear that in general the sum, or the average, of several Gompertz curves will not be a Gompertz curve; just as several logistics do not, in theory, give a logistic when added or when averaged. But just as it has been found in practice that the sum of a number of logistics does in fact often approximate closely a logistic as has been shown by Reed and Pearl (1927), it will be true that Gompertz curves will often add to give something very close to a Gompertz curve. And the general theory of averaging growth curves worked out by Merrell (1931) and applied by her to the logistic can be applied without modification to the Gompertz curve.

It may be further pointed out that the Gompertz curve may be generalized in the manner in which Pearl and Reed (1923) have generalized the logistic.

Pearl and Reed set

$$\frac{dy}{dx} = \frac{b}{k} y(k - y)f(x) \quad (8)$$

and on integration obtain

$$y = \frac{k}{1 + e^{F(x)}} \quad (9)$$

and assuming that $F(x)$ can be expanded as a Taylor's series, they reach

$$y = \frac{k}{1 + e^{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots}} \quad (10)$$

In a similar fashion, the differential equation of the Gompertz curve may be written

$$\frac{dy}{dx} = by (\log k - \log y), \quad (11)$$

and if we add an arbitrary function of time on the right hand side of this equation

$$\frac{dy}{dx} = by (\log k - \log y)f(x) \quad (12)$$

we have on integration

$$y = ke^{-e^{F(x)}} \quad (13)$$

and if $F(x)$ can be expanded in a Taylor's series we have

$$y = ke^{-e^{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots}} \quad (14)$$

It will be noted that if we wish to use only a finite number of terms in the power series, we must keep an odd power of x for our highest term, if our curve is to run from $y = 0$ to $y = k$. This restriction applies to both the generalized logistic and the generalized Gompertz curve.

We may rationalize the derivation of the Gompertz curve along the lines indicated by Ludwig (1929). Ludwig postulates that the relative growth rate $\frac{1}{y} \frac{dy}{dx}$ must decrease monotonically with continued growth. If now we write

$$\frac{1}{y} \frac{dy}{dt} = m - ny \quad (15)$$

we have the differential equation of the logistic. In this case the relative growth rate decreases as a linear function of growth already reached. If we set

$$\frac{1}{y} \frac{dy}{dx} = pe^{-qx} \quad (16)$$

we have on integration the Gompertz curve. In this case the relative growth rate decreases exponentially with time—in Gompertz's phrasing, in equal small intervals of time the organism loses equal proportions of its power to grow.

This rationalization of the Gompertz curve is that used by Wright (1926), Davidson (1928) and Weymouth, McMillin and Rich (1931). (Ludwig does not consider the Gompertz curve.) Other authors (citations in literature list) who have used the Gompertz curve as a growth curve have not attempted a rational derivation, but have apparently been led to its use more or less empirically, much as Pearl and Reed (1920) were originally led to the logistic.

Wright (1926) appears to have first suggested the use of the Gompertz curve for biological growth. He says:

"In organisms, on the other hand, the damping off of growth depends more on internal changes in the cells themselves, the process which Minot called cytomorphosis. The average growth power as measured by the percentage rate of increase tends to fall at a more or less uniform percentage rate, leading to asymmetrical types of *S*-shaped curves of which the form $\log \log \frac{k}{y} = a(b - x)$ is a simple example, instead of the logistic curve $\log \left(\frac{k}{y} - 1 \right) = a(b - x)$."

Following Wright, Davidson (1928) used the Gompertz curve to represent the growth in body weight of cattle. He pointed out that the ordinate

at the point of inflection is $\frac{1}{e}$ of the limiting ordinate. Davidson's figures actually cover only the upper part of the growth curve; his lowest weight is about 70 per cent of the limiting weight, so that no inflection appears in his actual data.

Weymouth, McMillin and Rich (1931) have used the Gompertz curve to represent the growth in shell size of the razor clam. They state that the curve also gives good fits for the guinea-pig and the rat. It must be noted that while their curves give good fits, they have found it necessary, in their most extensive series, to use two different curves to graduate the first and second halves of their data. They state:

"It is to be noticed, however, that after the eighth winter the slope is distinctly less and toward the end, due to inadequate data, the curve is irregular. It is true of all curves that, even before they become irregular because of small numbers, the curves deviate from the earlier trend, indicating that the growth is maintained at a higher rate than the first part of the curve would lead one to predict."

Weymouth and Thompson (1931) have also applied the Gompertz curve to the growth of the Pacific cockle.

It is of interest to compare this mode of deriving growth curves with that given by Lotka (1925). Lotka begins with the assumption that growth takes place under constant external conditions, so that the growth rate may be regarded as a function of attained growth alone:

$$\frac{dy}{dx} = F(y). \quad (17)$$

We now consider the equilibrium conditions, that is, the conditions where

$$\frac{dy}{dx} = F(y) = 0. \quad (18)$$

Clearly $F(y)$ must vanish for $y = 0$. Then if we assume that $F(y)$ can be expanded as a Taylor's series, the series will contain no constant term, and we shall have

$$\frac{dy}{dx} = ay + by^2 + cy^3 + \dots \quad (19)$$

We now note that (18) must have at least one other root, if there is to be some upper limit to growth. The simplest case which satisfies this condition is where the series in (19) terminates with the term in y^2 , so that

$$\frac{dy}{dx} = ay + by^2. \quad (20)$$

But this is the differential equation of the logistic. It does not appear that this particular line of reasoning is likely to lead to the Gompertz curve.

Summary.—The Gompertz curve and the logistic possess similar properties which make them useful for the empirical representation of growth phenomena. It does not appear that either curve has any substantial advantage over the other in the range of phenomena which it will fit. Each curve has three arbitrary constants, which correspond essentially to the upper asymptote, the time origin, and the time unit or "rate constant." In each curve, the degree of skewness, as measured by the relation of the ordinate at the point of inflection to the distance between the asymptote, is fixed. It has been found in practice that the logistic gives good fits on material showing an inflection about midway between the asymptotes. No such extended experience with the Gompertz curve is as yet available, but it seems reasonable to expect that it will give good fits on material showing an inflection when about 37 per cent of the total growth has been completed. Generalizations of both curves are possible, but here again there appears to be no reason to expect any marked difference in the additional freedom provided.

I should like to express my thanks to Dr. Raymond Pearl, at whose suggestion this paper was written, for his advice and criticism.

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REVERSIBLE COAGULATION IN LIVING TISSUE. X*

BY WILDER D. BANCROFT, ROBERT S. GUTSELL,¹ AND JOHN E. RUTZLER, JR.²

BAKER CHEMICAL LABORATORY, CORNELL UNIVERSITY

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The final test of the theory, that agglomeration accompanies addiction to morphine, is to show that a peptizing agent reduces the severity of withdrawal symptoms or else eliminates them entirely. Another phase of the problem is to show that the administration of a peptizing agent to an individual withdrawn from morphine stops post-withdrawal symptoms, particularly the craving for the drug. To this end a morphine addict was withdrawn from the drug while treating him with sodium rhodanate as the theory³ demands.

J. H., male, aged 49, a trained nurse, presented himself for treatment by us. He had been addicted to the use of morphine for 16 years, during which time he was withdrawn from the drug about six times. The methods of treatment used upon him were: insensible withdrawal, slow withdrawal, abrupt withdrawal and the Towns-Lambert treatment. Once the patient did not use morphine for four months after being withdrawn; other times he resumed the use of the drug in from two days to three or four weeks.

At the time that this treatment was started the patient was using 25 grains of morphine in two days, injecting it intravenously. An additional